SOLUTION **NOVEMBER 2015**

M. MATH LINEAR ALGEBRA **FINAL EXAM** **100 POINTS**

Notes:
(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.
(b) Assume only those results that have been proved in class. All other steps should be justified.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) All vector spaces are assumed to be finite dimensional, unless mentioned otherwise.

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1. [12 points] Let $T : V \to V$ be a linear map of vector spaces and let $W \subset V$ be a T-invariant subspace.
(i) Describe how T induces a natural linear map $\bar{T} : V/W \to V/W$.
(ii) Prove or disprove: If $T|_W$ is diagonalizable and $\bar{T}$ is diagonalizable then T is diagonalizable.

Solution: (i) As $W \subset V$, an element of V/W is an equivalence class $[v] = \{v + w : w \in W\}$. Now define a map $\bar{T} : V/W \to V/W$ by $\bar{T}[v] = [Tv]$. This map is well-defined as $[v_1] = [v_2] \implies v_1 - v_2 \in W \implies T(v_1 - v_2) \in W$(as W is T-invariant) $\implies Tv_1 - Tv_2 \in W \implies [Tv_1] = [Tv_2]$. Also $\bar{T}$ is linear as T is linear.
(ii) Consider a linear map $T : \mathbb{C}^2 \to \mathbb{C}^2$ by $T(1, 0) = (1, 0)$ and $T(0, 1) = (1, 1)$. Then the matrix representation of T w.r.t. standard ordered basis of $\mathbb{C}^2$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $W = \mathit{span}\{(1, 0)\}$ is a T-invariant subspace and $T|_W\ x = x, \forall\ x \in W$, which is certainly diagonalizable. Also $\mathbb{C}^2/W = \big\{[\alpha(0, 1)] : \alpha \in \mathbb{C}\big\}$. Then $\bar{T}[(0, 1)] = [T(0, 1)] = [(1, 1)] = [(0, 1)]$. Thus, $\bar{T}[y] = [y], \forall\ [y] \in \mathbb{C}^2/W$, and therefore it is diagonalizable. But T is not diagonalizable.

2. [12 points] Let F be a field, $X = (x_{ij})$ an $m \times n$ matrix over F, and let $a_1, \cdots, a_m, b_1, \cdots, b_n$ be elements of F. Let $Y = (y_{ij})$ be the $m \times n$ matrix given by $y_{ij} = x_{ij} + a_i + b_j$. Prove that $|\text{rank}(X) - \text{rank}(Y)| \le 2$.

Solution: Since $\text{rank}(A+B) \le \text{rank}(A)+\text{rank}(B), \forall\ A, B\ m\times n$ matrices, $\text{rank}(A) = \text{rank}(A-B+B) \le \text{rank}(A - B) + \text{rank}(B) \implies \text{rank}(A - B) \ge \text{rank}(A) - \text{rank}(B)$.

Therefore, $\text{rank}(X) - \text{rank}(Y) \le \text{rank}(X - Y) = \text{rank} \begin{pmatrix} -a_1 - b_1 & \cdots & -a_1 - b_n \\ \vdots & \vdots & \vdots \\ -a_m - b_1 & \cdots & -a_m - b_n \end{pmatrix}$

$$\le \text{rank} \begin{pmatrix} -a_1 & \cdots & -a_1 \\ \vdots & \vdots & \vdots \\ -a_m & \cdots & -a_m \end{pmatrix} + \text{rank} \begin{pmatrix} -b_1 & \cdots & -b_n \\ \vdots & \vdots & \vdots \\ -b_1 & \cdots & -b_n \end{pmatrix} = 2.$$

Similarly, $\text{rank}(Y) - \text{rank}(X) \le \text{rank}(Y - X) = \text{rank} \begin{pmatrix} a_1 + b_1 & \cdots & a_1 + b_n \\ \vdots & \vdots & \vdots \\ a_m + b_1 & \cdots & a_m + b_n \end{pmatrix} \le 2.$

Hence, $|\text{rank}(X) - \text{rank}(Y)| \le 2$.

3. [12 points] Let $V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$ be linear maps of vector spaces.
(i) Define what it means for the above sequence of maps to be exact.
(ii) If $ST = 0$, prove that $\text{rank}(S) + \text{rank}(T) \le \mathit{dim}(V_2)$.

Solution: (i) The sequence of maps $V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$ is to be exact if $im(T) = ker(S)$.
(ii) If $ST = 0$, implies $im(T) \subseteq ker(S) \implies \text{rank}(T) \leq \text{nullity}(S)$. Now by rank- nullity theorem for linear map $S : V_2 \to V_3$ we have $dim(V_2) = \text{rank}(S) + \text{nullity}(S)$. Therefore $\text{rank}(S) + \text{rank}(T) \leq \text{rank}(S) + \text{nullity}(S) = dim(V_2)$.

4. [12 points]
(i) Given unit vectors $u_1, \cdots, u_m$ in $\mathbb{R}^n$, show that there exists a pair u_i, u_j (with $i \neq j$) such that $u_i \cdot u_j \geq \frac{-1}{m-1}$.
(ii) Let $v_1, v_2, v_3 \in \mathbb{R}^n$ be nonzero vectors and let θ_{ij} be the angle between v_i and v_j. Show that $\min(\theta_{12}, \theta_{23}, \theta_{31}) \leq 2\pi/3$ with equality iff v_1, v_2, v_3 are coplanar with $\theta_{12} = \theta_{23} = \theta_{31}$.

Solution: (i) Let $v = u_1 + \cdots + u_m$. Then $v \cdot v \geq 0 \implies (u_1 + \cdots + u_m) \cdot (u_1 + \cdots + u_m) \geq 0 \implies \sum_{k=1}^m \sum_{l=1}^m u_k \cdot u_l \geq 0 \implies m + m(m-1) u_i \cdot u_j \geq 0$ (where $u_i \cdot u_j = \min\{u_k \cdot u_l : 1 \leq k, l \leq m, k \neq l\}$) $\implies u_i \cdot u_j \geq \frac{-1}{m-1}$.
(ii) Since, v_1, v_2, v_3 are nonzero vectors, let $u_i = v_i/|v_i|, 1 \leq i \leq 3$. Then θ_{ij} is the angle between u_i and u_j, $0 \leq \theta_{ij} \leq \pi$ and $u_i \cdot u_j = \cos(\theta_{ij})$. Therefore using above results we get, $\min(u_1 \cdot u_2, u_2 \cdot u_3, u_3 \cdot u_1) \geq \frac{-1}{2} \implies \min(\cos(\theta_{12}), \cos(\theta_{23}), \cos(\theta_{31})) \geq \frac{-1}{2} \implies \min(\theta_{12}, \theta_{23}, \theta_{31}) \leq 2\pi/3$.
Now if v_1, v_2, v_3 are coplanar with $\theta_{12} = \theta_{23} = \theta_{31}$, then $\theta_{12} + \theta_{23} + \theta_{31} = 2\pi \implies \min(\theta_{12}, \theta_{23}, \theta_{31}) = \theta_{12} = 2\pi/3$.
Let $\min(\theta_{12}, \theta_{23}, \theta_{31}) = 2\pi/3$. Then $\theta_{12} = \theta_{23} = \theta_{31} \implies u_i \cdot u_j = \frac{-1}{2}$, $\forall 1 \leq i, j \leq 3, i \neq j$ (where $u_i = v_i/|v_i|$) $\implies (u_1 + u_2 + u_3) \cdot (u_1 + u_2 + u_3) = 0 \implies u_1 + u_2 + u_3 = 0 \implies u_1, u_2, u_3$ are coplanar and so v_1, v_2, v_3 are coplanar. Also, if v_1, v_2, v_3 are coplanar, then $\theta_{12} + \theta_{23} + \theta_{31} = 2\pi$. Therefore if $\min(\theta_{12}, \theta_{23}, \theta_{31}) = 2\pi/3$, then $\theta_{12} = \theta_{23} = \theta_{31}$, otherwise $\theta_{12} + \theta_{23} + \theta_{31} > 2\pi$. Now, if v_1, v_2, v_3 are not coplanar and $\theta_{12} \neq \theta_{23}, \theta_{31}$, then for all $x, y \in \mathbb{R}$, $(xu_1 + yu_2 + u_3) \cdot (xu_1 + yu_2 + u_3) > 0 \implies x^2 + y^2 + 1 + 2xy\cos(\theta_{12}) + 2x\cos(\theta_{31}) + 2y\cos(\theta_{23}) > 0 \implies 3 + 2(\cos(\theta_{12}) + \cos(\theta_{31}) + \cos(\theta_{23})) > 0$ (letting $x = y = 1$) $\implies \cos(\theta_{12}) + \cos(\theta_{31}) + \cos(\theta_{23}) > \frac{-3}{2}$. Now let $\min(\theta_{12}, \theta_{23}, \theta_{31}) = \theta_{12}$, then $\theta_{23}, \theta_{31} \geq 2\pi/3$, as $\min(\theta_{12}, \theta_{23}, \theta_{31}) = 2\pi/3$. Therefore $\frac{-1}{2} \leq \cos(\theta_{31}), \cos(\theta_{23}) \leq -1$. Hence $\cos(\theta_{12}) + \cos(\theta_{31}) + \cos(\theta_{23}) > \frac{-3}{2} \implies \cos(\theta_{12}) > \frac{-1}{2} \implies \theta_{12} < 2\pi/3$, a contradiction.

5. [28 points] TRUE or FALSE. In each of the following statements, decide whether it is true or false and give brief explanations for your answer. You will get credit only if your explanation is correct.
(i) If A is a symmetric invertible matrix with coefficients in $\mathbb{C}$, then $A = Q^t Q$ for some invertible matrix Q over $\mathbb{C}$.
(ii) If A is Hermitian symmetric invertible matrix, then $A = Q^* Q$ for some invertible matrix Q over $\mathbb{C}$.
(iii) If A, B are real symmetric matrices having the same characteristic polynomial, then A is similar to B over $\mathbb{R}$.
(iv) If $\{v_i\}_{i=1}^n$ is an arbitrary basis of a Hermitain space $(V, < . >)$ and $T : V \to V$ is an isomorphism such that $< v_i, v_j > = < Tv_i, Tv_j >$ then T is unitary.

Solution:(i) This statement is TRUE. Let $A = LU$ be a LU-factorization of A with diagonal entries of L are 1 that is $l_{ii} = 1\ \forall i$. Now we show that if A is invertible then this factorization is unique. First of all, if A is invertible then both L and U are invertible as $det(A) = det(L)det(U)$. Now, let $A = \tilde{L}\tilde{U}$ be another factorization such that $\tilde{l}_{ii} = 1$, $\forall i$. Then $LU = \tilde{L}\tilde{U} \implies \tilde{L}^{-1}L = \tilde{U}U^{-1}$. The right hand side of this equation is a upper triangular matrix and left side is a lower triangular matrix. Therefore $\tilde{L}^{-1}L = \tilde{U}U^{-1} = D$, diagonal matrix. Since $l_{ii} = \tilde{l}_{ii} = 1 \implies \tilde{L}^{-1}L = I \implies L = \tilde{L}$. Therefore $U = \tilde{U}$. Now, since $A^t = A$ and A is invertible $\implies U^tL^t = LU \implies L = U^t \implies A = U^tU$, where U is an invertible matrix.
(ii)This statement is FALSE. Since, every eigenvalues of Q^*Q are positive whereas A can have negative

eigenvalues. Now to show every eigenvalues of Q^*Q is positive, let λ is an eigenvalue of Q^*Q, then $\lambda \neq 0$ as Q is invertible. Then there exists $x \in \mathbb{C}^n$ such that $Q^*Qx = \lambda x \implies \lambda < x, x >=< \lambda x, x >=< Q^*Qx, x >=< Qx, Qx >\geq 0 \implies \lambda > 0$.
(iii) This statement is TRUE. Since A and B are real symmetric matrices, they are diagonalizable. Now, since they also have same characteristic polynomial (and both are diagonalizable), their Jordan-Canonical form must be the same and hence they must be similar.
(iv) This statement is TRUE. Since for any $v, w \in V$,
$< T^*Tv, w >=< Tv, Tw >=< T(\sum_{i=1}^n a_i v_i), T(\sum_{j=1}^n b_j v_j) >=< \sum_{i=1}^n a_i Tv_i, \sum_{j=1}^n b_j Tv_j >$
$=< \sum_{i=1}^n \sum_{j=1}^n a_i \bar{b_j} < Tv_i, Tv_j >=< \sum_{i=1}^n \sum_{j=1}^n a_i \bar{b_j} < v_i, v_j >=< v, w > \implies T^*T = I$. Now since T is an isomorphism and $T^*T = I$ implies $T^* = T^{-1}$. Hence T is unitary.

6. [12 points] Classify upto similarity, all 5×5 matrices over $\mathbb{C}$ whose minimal polynomial is given $p(t) = (t+1)(t-1)^2$.

Solution: Since minimal polynomial and characteristic polynomial have same roots the minimal polynomial is given $p(t) = (t+1)(t-1)^2$ of 5×5 matrices over $\mathbb{C}$, the possible characteristic polynomial and their corresponding Jordan-Canonical form would be,

$$c_1(t) = (t+1)(t-1)^2(t+1)(t+1), \quad A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c_2(t) = (t+1)(t-1)^2(t+1)(t-1), \quad A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c_3(t) = (t+1)(t-1)^2(t-1)(t-1), \quad A_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c_4(t) = (t+1)(t-1)^2(t-1)^2, \quad A_4 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that there would be no matrices which consist a 2 Jordan block for eigenvalue -1, if so then minimal polynomial should have a factor of $(t+1)^2$.
Hence, upto similarity these are all 5×5 matrices over $\mathbb{C}$ whose minimal polynomial is given $p(t) = (t+1)(t-1)^2$.

7. [12 points] Let F be a field and let $a_0, \cdots, a_{n-1} \in F$. Give an example of an $n \times n$ matrix A over F such that $A^n + a_{n-1}A^{n-1} + \cdots + a_0$ equals to the zero matrix.

Solution: Consider a monic polynomial $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Let $\{v_1, \cdots, v_n\}$ be a ordered basis for F^n. Now consider a linear map $A : F^n \to F^n$ by $Av_i = v_{i+1}, \forall\ 1 \leq i \leq n-1, Av_n = -a_0v_1 - a_1v_2 - \cdots - a_{n-1}v_{n-1}$. Then the matrix representation of A w.r.t this ordered basis is

$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$. Now we have to show that $p(A)v = 0, \forall\ v \in F^n$, which will imply that $p(A)$ is zero matrix.

Since $v_k = A^{k-1}v_1, \forall\ 1 \leq k \leq n$, thus $p(A)v_k = A^{k-1}p(A)v_1$. Now $p(A)v_1 = A^n v_1 + a_{n-1}A^{n-1}v_1 + \cdots + a_0 v_1 = -a_0 v_1 - \cdots - a_{n-1}v_{n-1} + a_{n-1}v_{n-1} + \cdots + a_0 v_1 = 0$. Hence $p(A)v_k = A^{k-1}p(A)v_1 = 0,\ \forall\ 1 \leq k \leq n \implies p(A)v = 0, \forall\ v \in F^n$.